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SOME PROBLEMS OF LINEAR THERMOELASTICITY IN THE GINZBURG–LANDAU THEORY OF PHASE TRANSITIONS

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The well-posedness of the problem of linear thermoelasticity in the Ginzburg–Landau theory of phase transitions is proved.

Key words: *phase transitions, initial boundary-value problem, generalized solutions.*

Various mathematical models for description of phase transitions in multidimensional elastic media with the use of the nonconvex free-energy function were considered in [1]. In [2], the problem of [1] obtained by linearizing the equations of the Landau theory of phase transitions was studied. In the present paper, we consider the initial boundary-value problem for the system obtained in [1] by linearizing the equations of the Ginzburg–Landau theory of phase transitions.

Let Ω be a bounded domain of the n-dimensional space $\mathbb{R}^n$, $\partial\Omega \in C^3$, $Q_T = \Omega \times (0,T)$ for $T > 0$, $S_T = \partial\Omega \times (0,T)$ be the lateral surface of the cylinder Q_T, and Ω_τ be the cross section of Q_T formed by the plane $t = \tau$.

Problem 1. It is required to find the displacement vector $\boldsymbol{u}(\boldsymbol{x},t) = (u^{(1)}(\boldsymbol{x},t),\ldots,u^{(n)}(\boldsymbol{x},t))$ and temperature $\theta(\boldsymbol{x},t)$ that satisfy in Q_T the equations

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = -\alpha\nabla\theta + \mu\nabla\,\mathrm{div}\,\boldsymbol{u} - \gamma\nabla\Delta\,\mathrm{div}\,\boldsymbol{u} + \boldsymbol{f}; \tag{1}$$

$$\frac{\partial\theta}{\partial t} = k\Delta\theta - \beta\,\frac{\partial}{\partial t}(\mathrm{div}\,\boldsymbol{u}) + g, \tag{2}$$

subject to the initial conditions

$$\boldsymbol{u}\Big|_{\Omega_0} = \boldsymbol{u}_0(\boldsymbol{x}), \tag{3}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}\Big|_{\Omega_0} = \boldsymbol{u}_1(\boldsymbol{x}), \tag{4}$$

$$\theta\Big|_{\Omega_0} = \theta_0(\boldsymbol{x}) \tag{5}$$

and the boundary conditions

$$(\mathrm{div}\,\boldsymbol{u})\Big|_{S_T} = b_1(\boldsymbol{s},t), \qquad \Delta(\mathrm{div}\,\boldsymbol{u})\Big|_{S_T} = b_2(\boldsymbol{s},t), \qquad \theta\Big|_{S_T} = \theta_1(\boldsymbol{s},t).$$

Here $\mu > 0$, $\gamma > 0$, $k > 0$, α, and β are constants, $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},t)$, $g = g(\boldsymbol{x},t)$, $\boldsymbol{u}_0(\boldsymbol{x})$, $\boldsymbol{u}_1(\boldsymbol{x})$, $\theta_0(\boldsymbol{x})$, $b_1(\boldsymbol{s},t)$, $b_2(\boldsymbol{s},t)$, and $\theta_1(\boldsymbol{s},t)$ are specified functions, $\boldsymbol{x} = (x^{(1)},\ldots,x^{(n)})$ are the spatial variables, and t is the time. Since the case of inhomogeneous boundary conditions reduces to the case of homogeneous boundary conditions, we assume below that $b_1(\boldsymbol{s},t) = b_2(\boldsymbol{s},t) = \theta_1(\boldsymbol{s},t) = 0$.

We introduce the notation $v = \mathrm{div}\,\boldsymbol{u}$ and $\phi = \mathrm{div}\,\boldsymbol{f}$. We apply the divergence operator to the vector equation (1) and conditions (3) and (4). As a result, we obtain the following problem:

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Problem 1.1. It is required to find the functions $\theta(\boldsymbol{x},t)$ and $v(\boldsymbol{x},t)$ in Q_T that satisfy the equations

$$\frac{\partial^2 v}{\partial t^2} = -\alpha\Delta\theta + \mu\Delta v - \gamma\Delta^2 v + \phi, \tag{6}$$

$$\frac{\partial\theta}{\partial t} = k\Delta\theta - \beta\,\frac{\partial v}{\partial t} + g, \tag{7}$$

the initial conditions (5) and

$$v\Big|_{\Omega_0} = v_0(\boldsymbol{x}), \qquad v_0 = \operatorname{div}\boldsymbol{u}_0, \tag{8}$$

$$\frac{\partial v}{\partial t}\Big|_{\Omega_0} = v_1(\boldsymbol{x}), \qquad v_1 = \operatorname{div}\boldsymbol{u}_1, \tag{9}$$

and the boundary conditions

$$v\Big|_{S_T} = 0; \tag{10}$$

$$\Delta v\Big|_{S_T} = 0; \tag{11}$$

$$\theta\Big|_{S_T} = 0. \tag{12}$$

We assume that the functions θ and v are the classical solution of Problem 1.1. We multiply (6) by the function $\varphi \in C^2(\overline{Q}_T)$, which satisfies the condition

$$\varphi\Big|_{S_T} = 0, \qquad \varphi\Big|_{\Omega_T} = 0, \tag{13}$$

and integrate the resultant equality over the cylinder Q_T. Before doing this, we transform some integrals with allowance for the initial condition (9) and conditions (13):

$$\int\limits_{Q_T} v_{tt}\varphi\,dx\,dt = \int\limits_{Q_T}(v_t\varphi)_t\,dx\,dt - \int\limits_{Q_T} v_t\varphi_t\,dx\,dt = \int\limits_{\Omega_T} v_t\varphi\,dx - \int\limits_{\Omega_0} v_t\varphi\,dx - \int\limits_{Q_T} v_t\varphi_t\,dx\,dt = -\int\limits_{\Omega_0} v_1(x)\varphi(x,0)\,dx - \int\limits_{Q_T} v_t\varphi_t\,dx\,dt,$$

$$\int\limits_{Q_T}\varphi\Delta^2 v\,dx\,dt = \int\limits_{Q_T}\operatorname{div}(\varphi\nabla\Delta v)\,dx\,dt - \int\limits_{Q_T}\nabla\Delta v\nabla\varphi\,dx\,dt = \int\limits_{S_T}\frac{\partial(\Delta v)}{\partial\boldsymbol{n}}\,\varphi\,ds\,dt - \int\limits_{Q_T}\nabla\Delta v\nabla\varphi\,dx\,dt = -\int\limits_{Q_T}\nabla\Delta v\nabla\varphi\,dx\,dt.$$

As a result, we obtain

$$-\int\limits_{\Omega_0} v_1\varphi\,dx - \int\limits_{Q_T} v_t\varphi_t\,dx\,dt = \int\limits_{Q_T}(\phi - \alpha\Delta\theta)\varphi\,dx\,dt + \mu\int\limits_{Q_T}\Delta v\varphi\,dx\,dt + \gamma\int\limits_{Q_T}\nabla\Delta v\nabla\varphi\,dx\,dt.$$

Using the resultant identity, we introduce the concept of the generalized solution of Problem 1.1.

Definition 1. We call the pair of functions $\{\theta, v\}$, where $\theta \in W_2^{3,1}(Q_T)$ and $v \in W_2^{3,1}(Q_T)$, the generalized solution of Problem 1.1 if θ satisfies Eq. (7) almost everywhere in Q_T and conditions (5) and (12) and the function v satisfies the initial condition (8), the boundary conditions (10) and (11), and the identity

$$\int\limits_{Q_T}(\gamma\nabla\Delta v\nabla\varphi + \mu\Delta v\,\varphi + v_t\varphi_t)\,dx\,dt = -\int\limits_{\Omega_0} v_1\varphi\,dx - \int\limits_{Q_T}(\phi - \alpha\Delta\theta)\varphi\,dx\,dt \tag{14}$$

for all $\varphi \in W_2^1(Q_T)$ satisfying the conditions

$$\varphi\Big|_{S_T} = 0, \qquad \varphi\Big|_{\Omega_T} = 0. \tag{15}$$

Given the solution $\{\theta, v\}$ of Problem 1.1, one can determine the generalized solution of Problem 1 as $\{\theta, \boldsymbol{u}\}$, where $\boldsymbol{u}$ is found from the equation

$$\frac{\partial^2\boldsymbol{u}}{\partial t^2} = -\alpha\nabla\theta + \mu\nabla v - \gamma\nabla\Delta v + \boldsymbol{f}$$

with the known right side, subject to the initial conditions (3) and (4).

Theorem 1. *Let $\partial\Omega \in C^3$, $\phi, g \in W_2^1(Q_T)$, $\theta_0 \in W_2^3(\Omega)$, $v_0 \in W_2^3(\Omega)$, $v_1 \in W_2^1(\Omega)$, $\phi|_{S_T} = g|_{S_T} = 0$, and $\theta_0|_{\partial\Omega} = v_0|_{\partial\Omega} = \Delta v_0|_{\partial\Omega} = v_1|_{\partial\Omega} = 0$. Then, the generalized solution $\{\theta, v\}$ of Problem* 1.1 *exists and is unique. In this case, the following estimates are valid:*

$$\|\theta\|_{W_2^{3,1}(Q_T)} \leqslant C(\|\theta_0\|_{W_2^3(\Omega)} + \|v_0\|_{W_2^3(\Omega)} + \|v_1\|_{W_2^1(\Omega)} + \|g\|_{W_2^1(Q_T)} + \|\phi\|_{W_2^1(Q_T)}); \tag{16}$$

$$\|v\|_{W_2^{3,1}(Q_T)} \leqslant C(\|\theta_0\|_{W_2^3(\Omega)} + \|v_0\|_{W_2^3(\Omega)} + \|v_1\|_{W_2^1(\Omega)} + \|g\|_{W_2^1(Q_T)} + \|\phi\|_{W_2^1(Q_T)}) \tag{17}$$

(C are constants independent of θ_0, v_0, v_1, g, and ϕ, but C may depend on T).

Proof. To prove that the generalized solution of Problem 1.1 exists, we use the principle of contracting mappings. We construct the mapping

$$F\colon \quad M_T \to W_2^{3,1}(Q_T),$$

where $M_T = \{\theta \in W_2^{3,1}(Q_T)\colon\ \|\theta\|_{W_2^{3,1}(Q_T)} \leqslant m_0,\ \theta|_{\Omega_0} = \theta_0(\boldsymbol{x}), \text{ and } \theta|_{S_T} = 0\}$, which acts in the following manner. Given the function $\theta(\boldsymbol{x}, t) \in M_T$, we determine the function $v(\boldsymbol{x}, t) \in W_2^{3,1}(Q_T)$ that satisfies identity (14) and conditions (8), (10), and (11). Determining the function v, we find the function $\tilde{\theta}(\boldsymbol{x}, t) \in W_2^{3,1}(Q_T)$ that satisfies the equation

$$\tilde{\theta}_t = k\Delta\tilde{\theta} - \beta v_t + g$$

and the conditions $\tilde{\theta}\Big|_{\Omega_0} = \theta_0(\boldsymbol{x})$ and $\tilde{\theta}\Big|_{S_T} = 0$.

Let $\tilde{\theta} = F\langle\theta\rangle$. We show that, for reasonably small T, the mapping F is contracting and maps M_T into itself.

Lemma 1. *Let $\theta(\boldsymbol{x}, t) \in M_T$. Hence, the function v determined for a given function θ in constructing F satisfies the estimates*

$$\|v\|^2_{W_2^3(\Omega_t)} + \|v_t\|^2_{L_2(\Omega_t)} \leqslant c(\|v_0\|^2_{W_2^3(\Omega)} + \|v_1\|^2_{W_2^1(\Omega)}) + cT(\|\theta\|^2_{W_2^{3,1}(Q_T)} + \|\phi\|^2_{W_2^1(Q_T)}); \tag{18}$$

$$\|v\|^2_{W_2^{3,1}(Q_T)} \leqslant cT(\|v_0\|^2_{W_2^3(\Omega)} + \|v_1\|^2_{W_2^1(\Omega)}) + cT^2(\|\theta\|^2_{W_2^{3,1}(Q_T)} + \|\phi\|^2_{W_2^1(Q_T)}), \tag{19}$$

where c are constants independent of T, θ_0, v_0, v_1, and ϕ.

Proof. Let $u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), \ldots$ be a system orthonormalized in $L_2(\Omega)$ that contains all generalized eigenfunctions of the problem

$$\Delta u_k = \lambda_k u_k, \qquad \boldsymbol{x} \in \Omega, \quad u_k|_{\partial\Omega} = 0 \quad (k = 1, 2, \ldots).$$

The system $u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), \ldots$ is the orthonormalized basis in $L_2(\Omega)$, $\lambda_k < 0$ and $\lambda_k \to -\infty$ as $k \to \infty$ [3, p. 191]. The function $\Phi(\boldsymbol{x}, t) = \phi - \alpha\Delta\theta$ belongs to $W_2^1(Q_T)$. It follows from the Fubini theorem that $\Phi(\boldsymbol{x}, t) \in L_2(\Omega_t)$ for $t \in (0, T)$. For all $t \in (0, T)$, the functions $v_0(\boldsymbol{x}), v_1(\boldsymbol{x})$, and $\Phi(\boldsymbol{x}, t)$ can be expanded in the Fourier series in terms of the functions $u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), \ldots$:

$$v_0(\boldsymbol{x}) = \sum_{k=1}^{\infty} v_{0k} u_k(\boldsymbol{x}), \quad v_1(\boldsymbol{x}) = \sum_{k=1}^{\infty} v_{1k} u_k(\boldsymbol{x}), \quad \Phi(\boldsymbol{x}, t) = \sum_{k=1}^{\infty} \Phi_k(t) u_k(\boldsymbol{x}). \tag{20}$$

Here $v_{0k} = (v_0, u_k)_{L_2(\Omega)}$, $v_{1k} = (v_1, u_k)_{L_2(\Omega)}$, and $\Phi_k(t) = \int\limits_{\Omega} \Phi(\boldsymbol{x}, t) u_k(\boldsymbol{x})\, dx$.

Since $|\Phi_k(t)|^2 \leqslant \int\limits_{\Omega} \Phi^2(\boldsymbol{x}, t)\, dx \int\limits_{\Omega} u_k^2(\boldsymbol{x})\, dx = \int\limits_{\Omega} \Phi^2(\boldsymbol{x}, t)\, dx$, then $\Phi_k(t) \in L_2(0, T)$. According to the Parseval–Steklov equality, we obtain

$$\sum_{k=1}^{\infty} v_{0k}^2 = \|v_0\|^2_{L_2(\Omega)}, \qquad \sum_{k=1}^{\infty} v_{1k}^2 = \|v_1\|^2_{L_2(\Omega)} \tag{21}$$

and, for $t \in (0, T)$,

$$\sum_{k=1}^{\infty} \Phi_k^2(t) = \int\limits_{\Omega} \Phi^2(\boldsymbol{x}, t)\, dx.$$

Hence, we have

$$\sum_{k=1}^{\infty}\Phi_k^2(t)=\int\limits_{\Omega}\Phi^2(\boldsymbol{x},t)\,dx.$$

For the specified function $\theta\in M_T$, the solution v of problem (14), (8), (10), (11) can be written in the form of the series

$$v(\boldsymbol{x},t)=\sum_{k=1}^{\infty}U_k(t)u_k(\boldsymbol{x}),\tag{23}$$

where

$$U_k(t)=v_{0k}\cos\big(\sqrt{\gamma\lambda_k^2-\mu\lambda_k}\,t\big)+\frac{v_{1k}}{\sqrt{\gamma\lambda_k^2-\mu\lambda_k}}\sin\big(\sqrt{\gamma\lambda_k^2-\mu\lambda_k}\,t\big)$$

$$+\frac{1}{\sqrt{\gamma\lambda_k^2-\mu\lambda_k}}\int\limits_0^t\Phi_k(\tau)\sin\big(\sqrt{\gamma\lambda_k^2-\mu\lambda_k}\,(t-\tau)\big)\,d\tau.\tag{24}$$

Indeed, the function $U_k(t)$ belongs to $W_2^2(0,T)$, satisfies the initial conditions $U_k(0)=v_{0k}$ and $U_k'(0)=v_{1k}$ at $t=0$, and is a solution of the equation

$$U_k''+(\gamma\lambda_k^2-\mu\lambda_k)U_k=\Phi_k.$$

Since the function $V_k(\boldsymbol{x},t)=U_k(t)u_k(\boldsymbol{x})$ and any function $\varphi\in W_2^1(Q_T)$ that obeys conditions (15) satisfy the relations

$$\int\limits_{Q_T}\nabla\Delta V_k\nabla\varphi\,dx\,dt=\lambda_k\int\limits_{Q_T}\nabla V_k\nabla\varphi\,dx\,dt=-\lambda_k^2\int\limits_{Q_T}V_k\varphi\,dx\,dt,\qquad \mu\int\limits_{Q_T}\Delta V_k\varphi\,dx\,dt=\mu\lambda_k\int\limits_{Q_T}V_k\varphi\,dx\,dt,$$

$$\int\limits_{Q_T}V_{kt}\varphi_t\,dx\,dt=\int\limits_{\Omega}u_k(\boldsymbol{x})\Big(\int\limits_0^T U_k'(t)\varphi_t\,dt\Big)\,dx=\int\limits_{\Omega}u_k(\boldsymbol{x})\Big(-v_{1k}\varphi(\boldsymbol{x},0)-\int\limits_0^T U_k''(t)\varphi\,dt\Big)\,dx$$

$$=-v_{1k}\int\limits_{\Omega}u_k(\boldsymbol{x})\varphi(\boldsymbol{x},0)\,dx-\int\limits_{Q_T}u_k(\boldsymbol{x})\Phi_k\varphi\,dx\,dt+(\gamma\lambda_k^2-\mu\lambda_k)\int\limits_{Q_T}V_k\varphi\,dx\,dt,$$

the function $V_k(\boldsymbol{x},t)$ satisfies the integral identity

$$\int\limits_{Q_T}(\gamma\nabla\Delta V_k\nabla\varphi+\mu\Delta V_k\varphi+V_{kt}\varphi_t)\,dx\,dt=-v_{1k}\int\limits_{\Omega_0}u_k(\boldsymbol{x})\varphi\,dx-\int\limits_{Q_T}\Phi_k(t)u_k(\boldsymbol{x})\varphi\,dx\,dt.$$

Formula (24) implies that, for all $t\in[0,T]$, we have

$$|U_k(t)|\leqslant|v_{0k}|+|v_{1k}|\,|\gamma\lambda_k^2-\mu\lambda_k|^{-1/2}+|\gamma\lambda_k^2-\mu\lambda_k|^{-1/2}\int\limits_0^T|\Phi_k(t)|\,dt,$$

$$U_k^2(t)\leqslant 3\Big(v_{0k}^2+v_{1k}^2|\gamma\lambda_k^2-\mu\lambda_k|^{-1}+T|\gamma\lambda_k^2-\mu\lambda_k|^{-1}\int\limits_0^T\Phi_k^2(t)\,dt\Big),$$

$$\Big|\frac{dU_k}{dt}\Big|\leqslant|v_{0k}|\,|\gamma\lambda_k^2-\mu\lambda_k|^{1/2}+|v_{1k}|+\int\limits_0^T|\Phi_k(t)|\,dt,$$

$$\Big|\frac{dU_k}{dt}\Big|^2\leqslant 3\Big(v_{0k}^2|\gamma\lambda_k^2-\mu\lambda_k|+v_{1k}^2+T\int\limits_0^T\Phi_k^2(t)\,dt\Big).$$

Since, by the conditions of Theorem 1, the function $v_0 \in W_2^3(\Omega)$, $v_0\Big|_{\partial\Omega} = 0$, $\Delta v_0\Big|_{\partial\Omega} = 0$, its Fourier series (20) in the functions $\{u_k\}$ converges to it in the norm of the space $W_2^3(\Omega)$ [3, p. 253]. Similarly, the corresponding Fourier series for the functions v_1 and Φ converge to them in the norm of the space $W_2^1(\Omega)$. In this case, the following estimates are valid:

$$\sum_{k=1}^{\infty} v_{0k}^2|\lambda_k^3| \leqslant c\|v_0\|^2_{W_2^3(\Omega)}, \qquad \sum_{k=1}^{\infty} v_{1k}^2|\lambda_k| \leqslant c\|v_1\|^2_{W_2^1(\Omega)}, \qquad \sum_{k=1}^{\infty}|\lambda_k|\int_0^T \Phi_k^2\,dt \leqslant c\|\Phi\|^2_{W_2^1(Q_T)}. \tag{25}$$

We consider the partial sum of series (23)

$$v_N(\boldsymbol{x},t) = \sum_{k=1}^{N} U_k(t)u_k(\boldsymbol{x}).$$

For any $t \in [0,T]$, the function $v_N(\boldsymbol{x},t)$ and its derivative with respect to t belong to $W_2^3(\Omega)$, and the following estimates hold:

$$\begin{aligned}
\|v_N\|^2_{W_2^3(\Omega_t)} &= \Big\|\sum_{k=1}^{N} U_k(t)u_k(\boldsymbol{x})\Big\|^2_{W_2^3(\Omega_t)} = \int_\Omega \Big[\Big(\sum_{k=1}^{N} U_k(t)u_k(\boldsymbol{x})\Big)^2 + \Big(\sum_{i=1}^{n}\sum_{k=1}^{N} U_k(t)u_{kx_i}(\boldsymbol{x})\Big)^2 \\
&\quad + \Big(\sum_{i,j=1}^{n}\sum_{k=1}^{N} U_k(t)u_{kx_ix_j}(\boldsymbol{x})\Big)^2 + \Big(\sum_{i,j,l=1}^{n}\sum_{k=1}^{N} U_k(t)u_{kx_ix_jx_l}(\boldsymbol{x})\Big)^2\Big]\,dx \\
&\leqslant c\int_\Omega \Big(\sum_{i,j,l=1}^{n}\sum_{k=1}^{N} U_k(t)u_{kx_ix_jx_l}(\boldsymbol{x})\Big)^2\,dx \leqslant c\int_\Omega \sum_{i,j=1}^{n}\sum_{k=1}^{N} U_k^2(t)\Big|\frac{\partial(\nabla u_k(\boldsymbol{x}))}{\partial x_i\partial x_j}\Big|^2\,dx \\
&\leqslant c\int_\Omega \sum_{k=1}^{N} U_k^2(t)|\Delta\nabla u_k|^2\,dx \leqslant c\sum_{k=1}^{N} U_k^2(t)\lambda_k^2\int_\Omega |\nabla u_k|^2\,dx \qquad (26)\\
&= c\sum_{k=1}^{N} U_k^2(t)|\lambda_k|^3 \leqslant c\sum_{k=1}^{N}\Big(v_{0k}^2|\lambda_k|^3 + v_{1k}^2\frac{\lambda_k^2}{\gamma|\lambda_k-\mu/\gamma|} + T\frac{\lambda_k^2}{\gamma|\lambda_k-\mu/\gamma|}\int_0^T \Phi_k^2\,dt\Big) \\
&\leqslant c\sum_{k=1}^{N}\Big(v_{0k}^2|\lambda_k|^3 + v_{1k}^2|\lambda_k| + T|\lambda_k|\int_0^T \Phi_k^2\,dt\Big),
\end{aligned}$$

$$\Big\|\frac{\partial v_N}{\partial t}\Big\|^2_{L_2(\Omega_t)} = \Big\|\sum_{k=1}^{N} U_k'(t)u_k(\boldsymbol{x})\Big\|^2_{L_2(\Omega_t)} = \sum_{k=1}^{N}(U_k'(t))^2$$

$$\leqslant c\sum_{k=1}^{N}\Big(v_{0k}^2|\gamma\lambda_k^2-\mu\lambda_k| + v_{1k}^2 + T\int_0^T \Phi_k^2\,dt\Big) \leqslant c\sum_{k=1}^{N}\Big(v_{0k}^2(\gamma\lambda_k^2+\mu|\lambda_k|) + v_{1k}^2 + T\int_0^T \Phi_k^2\,dt\Big).$$

Passing to the limit in (26) as $N \to \infty$ and bearing in mind (21), (22), and (25), we obtain estimate (18) in Lemma 1. Estimate (19) follows from (18) after integration with respect to time from 0 to T.

Lemma 2. *Let $\theta(\boldsymbol{x},t) \in M_T$. Then, the following estimate is valid for the function $\tilde{\theta} = F\langle\theta\rangle$:*

$$\|\tilde{\theta}\|^2_{W_2^3(\Omega_t)} + \|\tilde{\theta}\|^2_{W_2^{3,1}(Q_T)} \leqslant c\mathrm{e}^{2c_1T}(T+1)(\|\theta_0\|^2_{W_2^3(\Omega)} + \|v\|^2_{W_2^{3,1}(Q_T)} + \|g\|^2_{W_2^1(Q_T)})$$

(c and c_1 are constants independent of T, θ_0, v_0, v_1, g, and ϕ).

Lemma 2 is a corollary of estimate (6.10) from [4, p. 207].

Lemma 3. *There exist a constant $m_0 > 0$ and a time $T_1 > 0$ such that F is a contracting mapping and maps M_{T_1} into itself.*

Proof. Let $\theta \in M_T$. Then, we have

$$\|\theta\|^2_{W_2^{3,1}(Q_T)} \leqslant m_0^2.$$

Lemmas 1 and 2 imply the estimate

$$\|\tilde{\theta}\|^2_{W_2^{3,1}(Q_T)} \leqslant c\mathrm{e}^{2c_1T}(T+1)(\|\theta_0\|^2_{W_2^3(\Omega)} + \|g\|^2_{W_2^1(Q_T)}$$

$$+ T(\|v_0\|^2_{W_2^3(\Omega)} + \|v_1\|^2_{W_2^1(\Omega)}) + T^2(m_0^2 + \|\phi\|^2_{W_2^1(Q_T)}))$$

$$\leqslant C_1(T)(\|\theta_0\|^2_{W_2^3(\Omega)} + \|g\|^2_{W_2^1(Q_T)} + \|v_0\|^2_{W_2^3(\Omega)} + \|v_1\|^2_{W_2^1(\Omega)} + \|\phi\|^2_{W_2^1(Q_T)}) + C_2(T)T^2m_0^2.$$

If we choose

$$m_0^2 > 2C_1(T)(\|\theta_0\|^2_{W_2^3(\Omega)} + \|v_0\|^2_{W_2^3(\Omega)} + \|v_1\|^2_{W_2^1(\Omega)} + \|g\|^2_{W_2^1(Q_T)} + \|\phi\|^2_{W_2^1(Q_T)}),$$

then, for reasonably small $T_1 > 0$, we can satisfy the estimate

$$\|\tilde{\theta}\|^2_{W_2^{3,1}(Q_{T_1})} \leqslant m_0^2,$$

i.e., the mapping F acts from M_{T_1} to M_{T_1}.

Let us show that the mapping F is contracting. Let $\theta^{(i)} \in M_{T_1}$, $i = 1, 2$. For the corresponding functions $v^{(i)}$ from Lemma 1, we obtain the estimate

$$\|v^{(1)} - v^{(2)}\|^2_{W_2^{3,1}(Q_{T_1})} \leqslant cT_1^2\|\theta^{(1)} - \theta^{(2)}\|^2_{W_2^{3,1}(Q_{T_1})}.$$

Lemma 2 yields the estimate

$$\|F\langle\theta^{(1)}\rangle - F\langle\theta^{(2)}\rangle\|^2_{W_2^{3,1}(Q_{T_1})} = \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|^2_{W_2^{3,1}(Q_{T_1})}$$

$$\leqslant c\mathrm{e}^{2c_1T_1}(T_1+1)\|v^{(1)} - v^{(2)}\|^2_{W_2^{3,1}(Q_{T_1})} \leqslant c\mathrm{e}^{2c_1T_1}(T_1+1)T_1^2\|\theta^{(1)} - \theta^{(2)}\|^2_{W_2^{3,1}(Q_{T_1})}.$$

If T_1 is chosen so that $c\mathrm{e}^{2c_1T_1}(T_1+1)T_1^2 = q < 1$, the mapping F is contracting.

By the Banach theorem on contracting mappings, the set M_{T_1} contains a single fixed point θ^*, which, together with the corresponding function v^*, is the solution of Problem 1.1 in the time interval $[0, T_1]$. The resulting solution can be continued to the intervals $[T_1, T_2]$, …, $[T_k, T_{k+1}]$, where $T_{k+1} - T_k > \delta > 0$ and δ does not depend on the number k. This follows from the estimates of Lemmas 1 and 2. Hence, the solution can be continued to any $T > 0$.

The uniqueness of the solution is proved using the rule of contradiction and estimates of Lemmas 1 and 2.

Estimate (16) follows from the fact that $\theta^*(\boldsymbol{x}, t) \in M_T$. Estimate (17) follows from (16) and Lemma 1. Theorem 1 is proved.

Remark 1. All the results can be extended to the case where the boundary conditions (10) and (12) are replaced by conditions of the form

$$\Big(\frac{\partial(\operatorname{div}\boldsymbol{u})}{\partial\boldsymbol{n}} + \sigma_1 \operatorname{div}\boldsymbol{u}\Big)\Big|_{S_T} = 0, \qquad \Big(\frac{\partial\theta}{\partial\boldsymbol{n}} + \sigma_2\theta\Big)\Big|_{S_T} = 0,$$

where σ_i are functions specified on S_T and $\boldsymbol{n}$ is the external normal to $\partial\Omega$.

Remark 2. For a smoother boundary of the domain $\partial\Omega$ and smoother data of the problem, one obtains smoother solutions, including the classical solution.

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